# Some Definability Results in Abstract Kummer Theory

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**Abstract**

Let $S$ be a semiabelian variety over an algebraically closed field, and let $X$ be an irreducible subvariety not contained in a translate of a proper algebraic subgroup of $S$. We show that the number of irreducible components of $[n]^{-1}(X)$ is bounded uniformly in $n$, and moreover that the bound is uniform in families $X_t$.

We prove this by Galois-theoretic methods. This proof allows for a purely model theoretic formulation, and applies in the more general context of divisible abelian groups of finite Morley rank. In this latter context, we deduce a definability result under the assumption of the Definable Multiplicity Property (DMP). We give sufficient conditions for finite Morley rank groups to have the DMP, and hence give examples where our definability result holds.

## 1 Introduction

Let $S$ be a semiabelian variety over an algebraically closed field of characteristic $p \geq 0$, and let $X$ be an irreducible subvariety. We say that $X$ is *Kummer-generic* (in $S$) if $[n]^{-1}(X)$ is irreducible for all $n$. Here, $[n] : S \to S$ denotes the multiplication-by-$n$ map.

It is easy to see that, since $S$ has Zariski-dense torsion, a necessary requirement for $X$ to be Kummer-generic is that it is not contained in any translate of a proper algebraic subgroup of $S$ — we call such $X$ *free*.

If $X$ is free, it does not follow that $X$ itself is Kummer-generic. For example, $X := \{(x, y) \mid y = (1 + x)^2\} \subseteq \mathbb{G}_m^2$ is irreducible and free, but $[2]^{-1}(X) = \{(x, y) \mid y^2 = (1 + x^2)^2\}$ is not irreducible, since $(y + 1 + x^2)(y - (1 + x^2)) = y^2 - (1 + x^2)^2$.

However, we prove:

**Theorem 1.1.** *Suppose $X \subseteq S$ is free. Then for some $n$, any irreducible component of $[n]^{-1}(X)$ is Kummer-generic in $S$.*

**Theorem 1.2.** *Suppose $S \to T$ is a parametrised family of semiabelian varieties and $X \subseteq S$ is a family of subvarieties. Then $\{t \mid X_t$ is Kummer-generic in $S_t\}$ is a constructible set.*

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Here, a family $S \to T$ of semiabelian varieties is assumed to have uniform group structure, i.e. a morphism $+ : S \times S \to S$ restricting to the group morphisms $+_t : S_t \times S_t \to S_t$ of the semiabelian varieties $S_t$.

In fact, we prove Theorem 1.2 by proving a uniform version of Theorem 1.1, Proposition 2.3 below.

Moreover, we prove generalisations of Theorems 1.1 and 1.2 (Theorems 6.4 and 6.7 respectively) in the context of divisible abelian groups of finite Morley rank. Theorems 1.1 and 1.2 may be easily derived from these more abstract counterparts, in fact with the slightly weaker assumption that the algebraic group be commutative and divisible. Nonetheless, we think it worth keeping the (purely algebraic) proofs of Theorems 1.1 and 1.2.

Versions of Theorem 1.1 have appeared previously in work on the model theory of universal covers of commutative algebraic groups, under the guise of "the $n = 1$ case of the Thumbtack Lemma". Zilber ([Zil06]) proves Theorem 1.1 in the case that $S = \mathbb{G}_m^n$ is an algebraic torus in characteristic 0, by consideration of divisors on a projective normal model of the function field of $X$; in [BZ11] it was noted that this argument goes through in positive characteristic. The second author ([Gav06, Gav09]) proved Theorem 1.1 for Abelian varieties in characteristic 0 by complex analytic and homotopic methods – in fact, the proof there proceeds by considering the fundamental group action on the universal covering space, and does not explicitly use the group structure on the abelian variety. The first author ([Bay09]) gave an alternative proof appealing to Lang-Néron's function field Mordell-Weil theorem.

Meanwhile, versions of Theorem 1.2 have arisen in the study of "green fields", fields expanded by a predicate for a generic multiplicative subgroup. In considering the theory of generic automorphisms of green fields, the third author ([Hil12]) found that he needed Theorem 1.2 for tori, and proved it by a finer consideration of divisors in the style of Zilber's proof of Theorem 1.1 for tori. Subsequently, it was observed by Roche that Theorem 1.2 for tori is a necessary part of the "collapse" of the green fields onto so-called *bad fields* by Baudisch, Martin Pizarro, Wagner and the third author ([BHMW09]).

The present work has corresponding applications. Indeed, in his thesis [Roc], Roche considers so-called *octarine fields*, certain expansions of abelian varieties by a predicate for a non-algebraic subgroup, a context which is similar to bad fields. In order to to be able to perform the "collapse" in this context, definability of Kummer-genericity (for abelian varieties in characteristic 0) is needed.

A case of Theorem 1.1, for hypersurfaces in tori in characteristic 0, appears in work of Ritt [Rit32, Section 8] in the context of factorising exponential sums.

The key idea of our proof is due to Ofer Gabber. In fact, he provided a proof (sketch) of Theorems 1.1 and 1.2, going via (étale) fundamental groups. We worked out the details and simplified his proof. In particular, we were able to extract the 'Galois theoretic' essence of the arguments and give proofs that do not use algebraic geometry in an essential way, replacing the use of the étale fundamental group of the variety by the absolute Galois group of the function field of the variety. Galois theory is available in a first-order structure ([Poi83]), and the statements and proofs transfer to this abstract model theoretic setting. Analysing the proofs, one sees that, in model theoretic terms, we use only that the complete theories $\mathrm{ACF}_p$ of algebraically closed fields have finite Morley rank and the Definable Multiplicity Property (DMP). Analysing this, we find that

we can prove analogous statements to Theorems 1.1 and 1.2 under more general conditions (partially even for type-definable groups of finite relative Morley rank); we do this in Section 6, transferring the relevant portions of Kummer theory to the more abstract setting.

First, in Section 4, we establish a criterion for a group $G$ of finite Morley rank (maybe with additional structure) to have the DMP which turns out to be very useful in practice. We show that $G$ has the DMP if and only if the generic automorphism is axiomatisable in $G$. Indeed, this equivalence holds for non-multidimensional theories with all dimensions strongly minimal. This generalises the case of strongly minimal theories which was proved by Hasson and Hrushovski [HH07].

These results apply to any group of finite Morley rank definable in the theory $\mathrm{DCF}_0$ of differentially closed fields of characteristic 0 (such groups are ([KP06, Proposition 2.4]) definably isomorphic to groups of the form $(G, s)^{\#} = \{x \in G \mid s(x) = \delta x\}$ for $(G, s)$ an algebraic D-group), as well as to groups definable in compact complex manifolds (which are known ([PS03], [Sca06]) to be definable extensions of definably compact groups by linear algebraic groups), and we thus obtain Theorems 1.1 and 1.2 in these contexts. In the differential case, we use that the generic automorphism is axiomatisable in $\mathrm{DCF}_0$ (a result of Hrushovski, see [Bus07]); in the compact complex analytic case, we obtain the axiomatisability of the generic automorphism using a result of Radin [Rad04] which asserts that irreducibility is definable in families.

Finally, we would like to mention an instance of the type-definable case. Let $A$ be an abelian variety which is defined over a non-perfect separably closed field $K$ of characteristic $p > 0$ (of finite degree of imperfection), and let $A^{\#}$ be the biggest $p$-divisible subgroup of $A(K)$ (equivalently, $A^{\#}$ may be defined as the smallest type-definable subgroup which is Zariski dense in $A$). Then Theorem 1.1 holds in $A^{\#}$ (for relatively definable subsets), since $A^{\#}$ has relative finite Morley rank ([BBP09]).

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# 2 Kummer theory

We first reformulate Kummer genericity as a condition on the image of the absolute Galois group of a function field on the product of the Tate modules.

Let $S$ be a semiabelian variety over an algebraically closed field. By standard results, $S$ is divisible.

Let $l \in \mathbb{N}$. The l-torsion $S[l]$ of $S$ is finite by l-divisibility of $S$ and dimension concerns. So $S[l]$ is a finite abelian group of exponent $l$, hence isomorphic to some $(\mathbb{Z}/l\mathbb{Z})^{k_l}$ if $l$ is a prime number.

Let $T = T_S$ be the inverse limit of the torsion

$$T = \varprojlim_n S[n]$$

with respect to the multiplication-by-m maps

$$[m] : S[mn] \to S[n].$$

Splitting into primary components, we have an isomorphism of profinite groups:

$$T \cong \Pi_l \mathbb{Z}_l^{k_l}$$

(Although we will not need this, in fact $k_l$ depends only on whether $l = p$: if $0 \to T \to S \to A \to 0$ is exact with $A$ an abelian variety and $T$ a torus, then $k_l = 2\dim(A) + \dim(T)$ if $l \neq p$, and $0 \leq k_p \leq \dim(A)$. It is worth noting that we need no assumption on $A$ being ordinary in order to prove our results.)

Let $X \subseteq S$ be an irreducible subvariety, let $K$ be algebraically closed such that $X$ and $S$ are defined over $K$, and let $b \in X$ be a generic point over $K$.

Let $G := \mathrm{Gal}(K(b)^{alg}/K(b))$.

The Kummer pairing provides a continuous homomorphism

$$\begin{aligned} \theta : G &\to T \\ \sigma &\mapsto (\sigma(b_n) - b_n)_n \end{aligned}$$

where the $b_n$ are arbitrary such that $mb_{nm} = b_n$ and $b_1 = b$; this is a well-defined homomorphism, since the torsion of $S$ is algebraic and hence contained in $S(K)$.

Let $Z = Z_X \leq T$ be the image of $\theta$. This does not depend on the choices of $K$ and $b$. It follows from continuity of $\theta$ that $Z$ is a closed subgroup of $T$.

Say $X$ is *n-Kummer-generic* in $S$ if and only if $[n]^{-1}(X)$ is irreducible.

**Lemma 2.1.** *$X$ is $n$-Kummer-generic in $S$ iff $Z + nT = T$; hence $X$ is Kummer-generic in $S$ iff $Z = T$.*

*Proof.* The map $[n]$ is closed - indeed, since $[n]$ generically has finite fibres, there exists an open $U \subseteq S$ such that $[n] \restriction_{[n]^{-1}(U)}$ is finite in the sense of algebraic geometry, and hence closed; covering $S$ with translates of $U$, we see that $[n]$ is closed.

It follows that some irreducible component $Y \subseteq [n]^{-1}X$ is such that $[n]Y = X$. If $Y'$ is another irreducible component, by considering its generic we see that $Y' \subseteq Y + \zeta \subseteq [n]^{-1}X$ for some $\zeta \in S[n]$; hence $Y' = Y + \zeta$. So $\{Y + \zeta | \zeta \in S[n]\}$ is the irreducible decomposition of $[n]^{-1}X$.

Let $b$ be generic in $X$ over $K$ and let $b_n \in Y$ be such that $nb_n = b$, so $b_n$ is generic in $Y$ over $K$. Then

$$\begin{aligned} & Z + nT = T \\ \Leftrightarrow\ & b_n \text{ is conjugate over } K \text{ to } b_n + \zeta \text{ for all } \zeta \in S[n] \\ \Leftrightarrow\ & Y = Y + \zeta \text{ for all } \zeta \in S[n] \\ \Leftrightarrow\ & [n]^{-1}X \text{ is irreducible.} \end{aligned}$$

The last equivalence is by our description of the decomposition. □

**Lemma 2.2.** *$X$ is Kummer-generic in $S$ iff $X$ is $l$-Kummer-generic in $S$ for all primes $l$.*

*Proof.* By Lemma 2.1, $X$ is $l$-Kummer-generic iff $Z + lT = T$; taking $l$-primary components $Z_l$ of $Z$ and $T_l$ of $T$, this is equivalent to $Z_l + lT_l = T_l$. But $T_l \cong \mathbb{Z}_l^{k_l}$, and $Z_l$ is a closed subgroup and hence a $\mathbb{Z}_l$-submodule, so $Z_l + lT_l = T_l$ iff $Z_l = T_l$ (to see this: note $Z_l + lT_l = T_l$ implies that, with respect to the isomorphism $T_l \cong \mathbb{Z}_l^{k_l}$, the submodule $Z_l$ contains the columns of a matrix equal modulo $l\mathbb{Z}_l$ to the identity matrix; but this matrix has determinant in $\mathbb{Z}_l^* = \mathbb{Z}_l \setminus l\mathbb{Z}_l$, hence is invertible in $\mathrm{Mat}_{k_l}(\mathbb{Z}_l)$). The lemma follows easily. □

In the next section, we prove:

**Proposition 2.3.** *Let $X_t \subseteq S_t$ be a family of free irreducible subvarieties of a family of semiabelian varieties. Then $Z_{X_t}$ is of finite index in $T_{S_t}$, and moreover this index is bounded.*

*Proof of Theorem 1.1 from Proposition 2.3.* By Proposition 2.3 with a constant family, $Z$ is of finite index in $T$, say $m$. So $m$ annihilates $T/Z$, hence $mT \leq Z$. If $Y$ is an irreducible component of $[m]^{-1}X$, it follows that $Z_Y = T$, i.e. that $Y$ is Kummer-generic in $S$. □

*Proof of Theorem 1.2 from Proposition 2.3.* It is clear that Kummer-genericity implies irreducibility. Irreducibility is a constructible condition - the set of $t$ for which $X_t$ is irreducible is constructible - so may assume that every $X_t$ in our family is irreducible.

Since the torsion of $S$ is Zariski-dense, Kummer-genericity implies freeness. Freeness is also a constructible condition – indeed, by Claim 1 below, $X \subseteq S$ is free iff the summation map $\Sigma : X^{2d} \to S$ is surjective, where $d = \dim(S)$. So we may assume that for all $t$, $X_t$ is free in $S_t$.

Irreducibility is a constructible condition, hence so is $l$-Kummer-genericity. By Lemma 2.2 and Proposition 2.3, for Kummer-genericity we need check only finitely primes $l$; hence, Kummer-genericity is also a constructible condition. □

# 3 Proof of Proposition 2.3

We prove this for a fixed $X \subseteq S$, and then note that the proof gives uniformity in families.

Let $d := \dim(S)$.

**Claim 1.** *The summation map*

$$\Sigma : X^d \to S$$

*is dominant.*

*Proof.* It suffices to show that if $k \leq d$ and the image of the $k$-ary summation map $X_k := \Sigma_1^k(X^k)$ has dimension $\dim(X_k) < d$, then $\dim(X_k + X) > \dim(X_k)$. Translating, we may assume $0 \in X$. By irreducibility it is then enough to see that, taking Zariski closures, $(X_k + X)^{\mathrm{Zar}} \neq X_k^{\mathrm{Zar}}$; but else, we would have $(X_k + X_k)^{\mathrm{Zar}} = X_k^{\mathrm{Zar}}$, whence $X_k^{\mathrm{Zar}}$ is a (proper) subgroup, contradicting freeness of $X$. □

Let $K$ be an algebraically closed field over which $X$ and $S$ are defined.

Let $a \in S$ be generic over $K$.

Let $\tilde{G} := \mathrm{Gal}(K(a)^{alg}/K(a))$.

Let $P$ be the set of (absolutely) irreducible components of $\Sigma^{-1}(a)$.

$\tilde{G}$ acts naturally on $P$. The action is transitive, since $X^d$ is irreducible and, by smoothness of $S$ and the dimension theorem, all irreducible components of $\Sigma^{-1}(a)$ have full dimension $\dim(X^d) - \dim(S)$.

Let $Z = Z_X \leq T_S = T$, defined as above.

**Claim 2.** *Let $V \in P$. The map $\tilde{\theta} : \tilde{G} \to T$*

$$\tilde{\theta} : \sigma \mapsto (\sigma(a_n) - a_n)_n$$

*(where $(a_n)$ is such that $ma_{nm} = a_n$ and $a_1 = a$) induces a group epimorphism*

$$\tilde{G} \to T/Z$$

*whose kernel contains $\mathrm{Stab}_{\tilde{G}}(V)$; hence the index of $Z$ in $T$ divides $|\tilde{G}/\mathrm{Stab}_{\tilde{G}}(V)| = |P|$, and in particular is finite.*

*Proof.* Obviously, $S$ is Kummer-generic in itself, as $[n]^{-1}(S) = S$ is irreducible for all $n$. Thus $Z_S = T$, showing that the map $\tilde{\theta}$ is surjective.

Now let $\overline{b}$ be generic in $V$ over $K(a)$. Note that $\overline{b}$ is generic in $X^d$ over $K$.

Let $\sigma \in \mathrm{Stab}_{\tilde{G}}(V) \leq \tilde{G}$. Then $\sigma$ extends to $\tau \in \mathrm{Gal}(K(a,\overline{b})^{alg}/K(a,\overline{b}))$. Choose $(\overline{b}_n)_{n\in\mathbb{N}}$ satisfying $m\overline{b}_{nm} = \overline{b}_n$ and $\overline{b}_1 = \overline{b}$. We then have

$$\begin{aligned}\tilde{\theta}(\sigma) &= (\sigma(\Sigma\overline{b}_n) - \Sigma\overline{b}_n)_n \\ &= (\Sigma(\tau\overline{b}_n - \overline{b}_n))_n \\ &= (\tau(\overline{b}_n)_1 - (\overline{b}_n)_1)_n + ... + (\tau(\overline{b}_n)_d - (\overline{b}_n)_d)_n\end{aligned}$$

But this is an element from $Z + \ldots + Z = Z$, and so $\tilde{\theta}$ induces a map as stated. □

This claim proves the non-uniform part of Proposition 2.3. Since the number of irreducible components of the generic fibre of $\Sigma$ is bounded uniformly in families, we obtain Proposition 2.3.

In our initial proof, we used the following remark to reduce the statement to the case of curves. This is no longer needed, and so we include it without proof. (One may use arguments similar to [Poi01, Proof of Lemme 3.1].)

*Remark* 3.1. Let $X \subseteq S$ be a Kummer-generic variety of dimension at least 2. Let $H \subseteq S$ be a generic hyperplane (with respect to some embedding of $S$ into some $\mathbb{P}^n$). Then $X \cap H$ is Kummer-generic.

We would also like to remark that the proof of Proposition 2.3 applies generally to endomorphisms of abelian algebraic groups with finite kernel, in particular to the Artin-Schreier map $\wp : \mathbb{G}_a^n \to \mathbb{G}_a^n$; $\overline{x} \mapsto (x_i^p - x_i)_i$ in characteristic $p \neq 0$, and its compositional iterates $\wp^{(m)}$. We therefore obtain an analogue of Theorem 1.1:

*Remark* 3.2. Say $X \subseteq \mathbb{G}_a^n$ is *Artin-Schreier-generic* iff $(\wp^{(m)})^{-1}X$ is irreducible for all $m$. So if $X \subseteq \mathbb{G}_a^n$ is free in the sense that the summation map $\Sigma : X^n \to \mathbb{G}_a^n$ is dominant, then for some $m$ each irreducible component of $(\wp^{(m)})^{-1}X$ is Artin-Schreier-generic.

# 4 DMP in groups of finite Morley rank

The aim of the rest of this paper is to give model theoretic generalisations of Theorems 1.1 and 1.2. We use standard model theoretic terminology and concepts; see e.g. [Pil96] for these.

We show here a number of results which will be useful when generalising our results for semiabelian varieties to groups of finite Morley rank.

In this and the following section, we will often write $x, y, \ldots$ for (finite) tuples of variables; similarly, $a, b, \ldots$ may denote tuples of elements. The Morley rank of a (partial) type $\pi$ will be denoted $\mathrm{RM}(\pi)$, its Morley degree by $\mathrm{DM}(\pi)$. Moreover, we write $\mathrm{RDM}(\pi)$ for the pair $(\mathrm{RM}(\pi), \mathrm{DM}(\pi))$.

**Definition 4.1.** We say that the theory $T$ is *almost $\aleph_1$-categorical* if $T$ is non-multidimensional, with all dimensions strongly minimal, i.e. there is a fixed set of strongly minimal sets $\{D_i \mid i \in I\}$ (in $T^{eq}$) such that every non-algebraic type is non-orthogonal to one of the $D_i$'s.

Note that in the previous definition we do not assume that the language is countable.

Observe that $T$ is almost $\aleph_1$-categorical iff $T^{eq}$ is. Moreover, in an almost $\aleph_1$-categorical theory, there are only finitely many non-orthogonality classes of strongly minimal types, so we may assume that $I$ is finite. For background on almost $\aleph_1$-categorical theories, in particular a proof of the following fact, we refer to [PP02].

**Fact 4.2.** *Let $T$ be almost $\aleph_1$-categorical.*
*Then, in $T^{eq}$, Morley rank is finite, definable and equal to Lascar* U*-rank.*

Let $T$ be a theory of finite Morley rank. Recall the following definitions:

- Morley rank is *definable* in $T$ if for every formula $\varphi(x, y)$ and every $r \in \mathbb{N}$ there is a formula $\theta(y)$ such that $\mathrm{RM}(\varphi(x, a)) = r$ iff $\models \theta(a)$.
- $T$ has the *definable multiplicity property* (DMP) if for every formula $\varphi(x, y)$ and every $(r, d) \in \mathbb{N}^2$ there is a formula $\theta(y)$ such that $\mathrm{RDM}(\varphi(x, a)) = (r, d)$ iff $\models \theta(a)$.

To establish the DMP, by compactness, it is enough to show that whenever $\mathrm{RDM}(\varphi(x, a)) = (r, d)$, there is $\theta(y) \in \mathrm{tp}(a)$ such that $\mathrm{RDM}(\varphi(x, a')) = (r, d)$ whenever $\models \theta(a')$.

*Remark* 4.3. Almost $\aleph_1$-categorical theories were introduced in [Bel73]; though stated in terms of extensions of models, his definition is quite close to the one we use. These theories were further studied by [Eri75] who in particular proved the finiteness of Morley rank.

**Fact 4.4** (Lascar [Las85])**.** *Let $G$ be a group of finite Morley rank and $T = \mathrm{Th}(G)$. Then $T$ is almost $\aleph_1$-categorical. In particular, Morley rank is finite, definable and equal to Lascar* U*-rank.*

**Definition 4.5.** Let $p(x)$ be a stationary type. Then $p$ is said to be *uniformly of Morley degree 1* if there is a formula $\varphi(x, a) \in p$ and $\theta(y) \in \mathrm{tp}(a)$ such that $p$ is the unique generic type in $\varphi(x, a)$ and $\mathrm{DM}(\varphi(x, a')) = 1$ whenever $\models \theta(a')$.

Note that if Morley rank is definable in $T$ and $p$ is uniformly of Morley degree 1, we may witness this by formulas $\varphi$ and $\theta$ such that $\mathrm{RM}(\varphi(x, a')) = \mathrm{RM}(p)$ whenever $\models \theta(a')$.

For $p, q \in S(B)$, we say that $p$ is a *finite cover* of $q$ if there are $a \models p$ and $b \models q$ such that $b \in \mathrm{dcl}(Ba)$ and $a \in \mathrm{acl}(Bb)$. The proof of the following lemma is easy and left to the reader.

**Lemma 4.6.** *Assume that Morley rank is finite and definable in $T$, and let $p$ be uniformly of Morley degree 1.*

1. *If $q$ is parallel to $p$, then $q$ is uniformly of Morley degree 1.*
2. *If $p$ is a finite cover of $q$, then $q$ is uniformly of Morley degree 1.*
3. *$T$ has the DMP iff every stationary type is uniformly of Morley degree 1.* □

**Lemma 4.7.** *Assume that Morley rank is finite, definable and equal to* U*-rank in $T = T^{eq}$. Then $T$ has the DMP iff every strongly minimal type is uniformly of Morley degree 1.*

*Proof.* For $T$ strongly minimal, this is shown in [Hru92].

The condition is clearly necessary. In order to show that it is sufficient, it is enough to show that any stationary type $p$ is uniformly of Morley degree 1, by Lemma 4.6(3). We prove this by induction on $\mathrm{RM}(p) = r$, the case $r = 1$ being true by assumption (and $r = 0$ being trivial).

Assume the result is true for types of Morley rank $\leq r$ and let $p(x) \in S(B)$ be stationary, $\mathrm{RM}(p) = r + 1$.

Since $0 < \mathrm{U}(p) < \omega$, there is a minimal type $p_1(x_1)$ (i.e. $\mathrm{U}(p_1) = 1$) such that $p \not\perp p_1$ (see [Pil96, Lemma 2.5.1]). By Lemma 4.6(1), we may replace $p$ and $p_1$ by non-forking extensions and thus assume that $p, p_1 \in S(M)$ for some model $M$ and $p \not\perp^a p_1$, whereby there are $a \models p$ and $a_1 \models p_1$ such that $a_1 \in \mathrm{acl}(Ma)$. By Lemma 4.6(2), we may replace $p$ by the finite cover $\mathrm{tp}(aa_1/M)$ and thus assume that $p_1$ is given by the restriction of $p(x)$ to a subtuple $x_1$ of the variables. Let $a_2$ be such that $a = a_1a_2$. By assumption, $\mathrm{U} = \mathrm{RM}$, so $p_1$ is strongly minimal. Let $q(x_2) = \mathrm{tp}(a_2/Ma_1)$. By $\omega$-stability, $\mathrm{tp}(a_2/M\tilde{a}_1)$ is stationary for some finite $\tilde{a}_1$ with $a_1 \subseteq \tilde{a}_1 \subseteq \mathrm{acl}(Ma_1)$. Enlarging $a_1$ if necessary, we may thus assume that $q$ is stationary.

Moreover, the Lascar inequalities and $\mathrm{RM} = \mathrm{U}$ imply that $\mathrm{RM}(q) = r$.

Using definability of Morley rank and the induction hypothesis, we may find an $\mathcal{L}$-formula $\varphi(x_2, x_1, z)$, $b \in M$ and $\theta(z) \in \mathrm{tp}(b)$ such that

(i) $p$ is the unique generic type in $\varphi(x_2, x_1, b)$ over $M$;

(ii) $\mathrm{RM}(\varphi(x_2, x_1, b')) = r + 1$ whenever $\models \theta(b')$;

(iii) $\psi(x_1, b') = \exists x_2 \varphi(x_2, x_1, b')$ is strongly minimal whenever $\models \theta(b')$ (in particular, $p_1$ is the unique generic type in $\psi(x_1, b)$ over $M$);

(iv) whenever $\varphi(x_2, a_1', b')$ is consistent, $\mathrm{RDM}(\varphi(x_2, a_1', b')) = (r, 1)$.

It is routine to check that $\varphi(x_2, x_1, b)$ and $\theta(z)$ witness that $p$ is uniformly of Morley degree 1. □

It follows in particular from the previous lemma that in any strongly minimal theory without the DMP, there is a strongly minimal type $p$ which is not uniformly of Morley degree 1. To illustrate this, let us recall the following example due to Hrushovski [Hru92].

*Example* 1. Let $V$ be a non trivial vector space over $\mathbb{Q}$ and $0 \neq v_0 \in V$. Let $D = V \times \{0, 1\}$ equipped with the projection $\pi : D \to V$ and the function $f : D \to D$, $f(v, i) = (v + v_0, i)$. Then $T = \mathrm{Th}(D)$ is strongly minimal without the DMP. For any $b \in D$, the formula $\varphi(x, y, b)$ given by $\pi(x) = \pi(y) + \pi(b)$ is of Morley rank 1, and it is strongly minimal iff $\pi(b) \notin \mathbb{Z} \cdot v_0$. (For $b \in \mathbb{Z} \cdot v_0$, one has $\mathrm{DM}(\varphi(x, y, b)) = 2$.)

Thus, for generic $b$ the (unique) generic type $p(x, y)$ of $\varphi(x, y, b)$ is strongly minimal and not uniformly of Morley degree 1. In particular, if $q(x)$ denotes the generic 1-type in $T$, $p$ and $q$ are non-orthogonal strongly minimal types, $q$ is uniformly of Morley degree 1 and $p$ is not.

**Lemma 4.8.** *Let $T$ be almost $\aleph_1$-categorical, and let $\varphi(x, z)$ be a formula. There is $N \in \mathbb{N}$ such that $\mathrm{DM}(\varphi(x, b)) \leq N$ for all $b$.*

*Proof.* Adding parameters to the language if necessary, we may assume that there are strongly minimal sets $(D_i)_{1 \leq i \leq k}$ defined over $\emptyset$ such that every non-algebraic type is non-orthogonal to one of the $D_i$'s.

**Claim.** *Every global type $p(x)$ is a generic type in some $\varphi(x, b)$, where $\varphi(x, z)$ is an $\mathcal{L}$-formula for which the statement of the lemma is true.*

We prove the claim by induction on $r = \mathrm{RM}(p)$, the case $r = 0$ being trivial.

Let $p$ be strongly minimal. By assumption, $p \not\perp D_i$ for some $i$. Let $D_i$ be defined by $\psi_i(y)$. Since $p(x)$ is a global type, there is $\varphi(x, b) \in p$ and a finite-to-finite correspondence $\chi(x, y, c)$ between $\varphi(x, b)$ and $\psi_i(y)$. Let $m, n \in \mathbb{N}$ such that $\chi(x, y, c)$ generically induces an $m$-to-$n$ correspondence (i.e. outside a finite set). We may assume that $b = c$ and that $\chi(x, y, b')$ generically induces an $m$-to-$n$ correspondence between $\varphi(x, b')$ and $\psi_i(y)$ whenever $\varphi(x, b')$ is consistent. Since $\psi_i(y)$ is strongly minimal, $\mathrm{DM}(\varphi(x, b')) \leq m$ for all $b'$. This proves the case $r = 1$.

For the induction step, we argue as in the proof of Lemma 4.7. Let $\mathrm{RM}(p) = r + 1$. Replacing $p$ by a finite cover, we may assume that $p(x) = p(x_1 x_2)$ and, by induction, that there is an $\mathcal{L}$-formula $\varphi(x_2, x_1, z)$ and $b \in M$ such that

- $p$ is the unique generic type in $\varphi(x_2, x_1, b)$ over $M$;
- $\mathrm{RM}(\varphi(x_2, x_1, b')) = r + 1$ whenever it is consistent;
- $\psi(x_1, b') = \exists x_2 \varphi(x_2, x_1, b')$ is of Morley rank 1 and degree $\leq N_1$ whenever it is consistent;
- whenever $\varphi(x_2, a_1', b')$ is consistent, it is of Morley rank $r$ and degree $\leq N_2$.

Then $\varphi(x_2, x_1, b')$ is of Morley degree $\leq N_1 N_2$ for every $b'$, so the claim is proved.

Now let $\varphi(x, z)$ be given. For a parameter $b$ with $\mathrm{DM}(\varphi(x, b)) = d$, let $p_1, \ldots, p_d$ be the (global) generic types of $\varphi(x, b)$. By the claim, for $1 \leq i \leq d$, there exists a formula $\chi(x, z_i)$, a parameter $b_i$ and $N_i \in \mathbb{N}$ such that $p_i$ is

generic in $\chi_i(x, b_i)$ and $\mathrm{DM}(\varphi(x, b'_i)) \leq N_i$ for every $b'_i$. Since Morley rank is definable, there is $\theta(y) \in \mathrm{tp}(b)$ such that whenever $\models \theta(b')$, there are $b'_1, \ldots, b'_d$ with $\mathrm{RM}\left(\varphi(x, b') \wedge \neg \bigvee_{i=1}^{d} \chi_i(x, b'_i)\right) < \mathrm{RM}(\varphi(x, b')) = \mathrm{RM}(\chi_i(x, b'_i))$ for $i = 1, \ldots, d$. So $\mathrm{DM}(\varphi(x, b')) \leq \sum_{i=1}^{d} N_i$ if $\models \theta(b')$.

Since such a formula exists for every $b$, we are done by compactness. $\square$

Now let $T$ be a theory which is complete and has quantifier elimination in some language $\mathcal{L}$, and let $\sigma \notin \mathcal{L}$ be a new unary function symbol. Consider $T_\sigma := T \cup \{\sigma \text{ is an } \mathcal{L}\text{-automorphism}\}$, a theory in the language $\mathcal{L} \cup \{\sigma\}$. Denote by $TA$ the model companion of $T_\sigma$ if it exists.

In the proof of the following result, we proceed as in [HH07], where Corollary 4.11 is shown for $T$ strongly minimal.

**Proposition 4.9.** *Let $T$ be stable, and assume $TA$ exists. Then, for strongly minimal types in $T$, being uniformly of Morley degree 1 is invariant under non-orthogonality.*

*Proof.* Let $p, q$ be strongly minimal types such that $p \not\perp q$ and $q$ is uniformly of Morley degree 1. We have to show that $p$ is uniformly of Morley degree 1. As in the proof of Lemma 4.7, using Lemma 4.6, we may assume that $p = \mathrm{tp}(\beta/M)$ is a finite cover of $q = \mathrm{tp}(b/M)$, and even that $b$ is a subtuple of $\beta$.

Let $\beta_1 = \beta, \beta_2, \ldots, \beta_n$ be the $Mb$-conjugates of $\beta$, and put $\overline{\beta}_1 = (\beta_1, \ldots, \beta_n)$. We may assume that $p = \mathrm{tp}(\overline{\beta}_1/M)$. Then, if $\overline{\beta}_1, \ldots, \overline{\beta}_r$ are the $Mb$-conjugates of $\overline{\beta}_1$, any $\overline{\beta}_i$ is of the form

$$\overline{\beta}_i = (\beta_{\tau_i(1)}, \ldots \beta_{\tau_i(n)})$$

for some $\tau_i \in S_n$. Moreover, by construction,

$$G := \{\tau_i \mid 1 \leq i \leq r\} \leq S_n$$

is a subgroup of $S_n$.

Since $q$ is uniformly of Morley degree 1 by assumption, there are $\mathcal{L}$-formulas $\tilde{\varphi}(\overline{x}, y)$, $\varphi(x, y)$, a parameter $a \in M$, a formula $\theta(y) \in r(y) = \mathrm{tp}(a)$, and a definable function $\pi$ from the $\overline{x}$-sort to the $x$-sort satisfying the following properties:

(i) $\tilde{\varphi}(\overline{x}, a)$ and $\varphi(x, a)$ are strongly minimal, with generic types $p$ and $q$, respectively;

(ii) whenever $\models \theta(a')$, $\varphi(x, a')$ is strongly minimal, and $\pi$ defines a surjection from $\tilde{\varphi}(\overline{x}, a')$ onto $\varphi(x, a')$;

(iii) whenever $\models \theta(a')$, denoting $\tilde{D}_{a'}$ and $D_{a'}$ the sets defined by $\tilde{\varphi}(\overline{x}, a')$ and $\varphi(x, a')$, resp., all fibres of the map $\pi : \tilde{D}_{a'} \twoheadrightarrow D_{a'}$ are regular $G$-orbits for the definable action of $G$ (by permutation) on the $\overline{x}$-sort, i.e. if $b' \in D_{a'}$, then $|\pi^{-1}(b')| = |G|$ and if $\pi^{-1}(b') = \{\overline{\beta}'_1, \ldots, \overline{\beta}'_s\}$, letting $\overline{\beta}'_1 = (\beta'_1, \ldots, \beta'_n)$, any $\overline{\beta}'_i$ is of the form $(\beta'_{\tau(1)}, \ldots, \beta'_{\tau(n)})$ for some $\tau \in G$.

**Claim.** *For any $\tau \in G$, there is an $\mathcal{L}$-formula $\theta_\tau(y) \in r(y)$ implying $\theta(y)$ such that, in $TA$, the following implication holds:*

$$\theta_\tau(y) \wedge \sigma(y) = y \vdash \exists x\, \exists \overline{x} \left[\pi(\overline{x}) = x \wedge \sigma(x) = x \wedge \tilde{\varphi}(\overline{x}, y) \wedge \bigwedge_{i=1}^{n} \sigma(x_i) = x_{\tau(i)}\right]$$

*Proof of the claim.* By compactness, it is enough to show that

$$r(y) \cup \{\sigma(y) = y\} \vdash \exists x\, \exists \overline{x} \left[ \pi(\overline{x}) = x \wedge \sigma(x) = x \wedge \tilde{\varphi}(\overline{x}, y) \wedge \bigwedge_{i=1}^{n} \sigma(x_i) = x_{\tau(i)} \right].$$

Let $(M', \sigma) \models TA$ and $a' \in M'$ such that $\sigma(a') = a'$ and $\models r(a')$. Then, $\tilde{D}_{a'}$ is strongly minimal (in $T$). Let $\overline{\beta}' = (\beta'_1, \ldots, \beta'_n)$ be generic in $\tilde{D}_{a'}$ over $M'$. Then $\tau \cdot \overline{\beta}' := (\beta_{\tau(1)}, \ldots, \beta_{\tau(n)})$ is also generic in $\tilde{D}_{a'}$ over $M'$. Thus, $\mathrm{tp}_{\mathcal{L}}(\overline{\beta}'/M') = \mathrm{tp}_{\mathcal{L}}(\tau \cdot \overline{\beta}'/M')$. Since $\sigma(a') = a'$, it follows that $\sigma \restriction_{\mathrm{acl}(a')} \cup \{\beta'_i \mapsto \beta'_{\tau(i)} \mid 1 \leq i \leq s\}$ is an elementary map. So, in some elementary extension of $(M', \sigma)$, there is such a tuple $\overline{\beta}'$ such that $\sigma(\beta'_i) = \beta'_{\tau(i)}$ for all $i$. Moreover, by construction, $\pi(\overline{\beta}') = \pi(\sigma(\overline{\beta}'))$, so $\sigma(\pi(\overline{\beta}')) = \pi(\overline{\beta}')$. This proves the claim. $\square$

Let $\tilde{\theta}(y) = \bigwedge_{\tau \in G} \theta_\tau(y)$. We will show that $\tilde{\varphi}(\overline{x}, a')$ is strongly minimal whenever $\models \tilde{\theta}(a')$. This will finish the proof.

From now on, we may proceed exactly as in [HH07].

By (ii), $\mathrm{RM}(\tilde{D}_{a'}) = 1$. Let $\overline{\delta}$ and $\overline{\delta}'$ be generic in $\tilde{D}_{a'}$ over $a'$. We have to show that $\mathrm{stp}(\overline{\delta}/a') = \mathrm{stp}(\overline{\delta}'/a')$. Since $D_{a'}$ is strongly minimal by (i), $\mathrm{stp}(\pi(\overline{\delta})/a') = \mathrm{stp}(\pi(\overline{\delta}')/a')$, so we may assume that $\pi(\overline{\delta}) = \pi(\overline{\delta}') = d$.

By (iii), $G$ acts regularly on $\pi^{-1}(d)$, via

$$\tau \cdot \overline{\delta}'' = \tau \cdot (\delta''_1, \ldots, \delta''_n) = (\delta''_{\tau(1)}, \ldots, \delta''_{\tau(n)}).$$

In particular, there is $\rho \in G$ such that $\rho \cdot \overline{\delta} = \overline{\delta}'$. Consider

$$H := \left\{ \tau \in G \mid \mathrm{id} \restriction_{\mathrm{acl}(a')} \cup \{\overline{\delta} \mapsto \tau \cdot \overline{\delta}\} \text{ is an elementary map} \right\}.$$

Note that $\tau$ is in $H$ iff there is an element $\rho \in G$ such that $\mathrm{id} \restriction_{\mathrm{acl}(a')} \cup \{\rho \cdot \overline{\delta} \mapsto (\tau\rho) \cdot \overline{\delta}\}$ is an elementary map iff $\mathrm{id} \restriction_{\mathrm{acl}(a')} \cup \{\rho \cdot \overline{\delta} \mapsto (\tau\rho) \cdot \overline{\delta} \mid \rho \in G\}$ is an elementary map. In particular, $H \leq G$ is a subgroup.

Let $\mathrm{id} \neq \tau \in G$. We may embed $(\mathrm{acl}(a'), \mathrm{id})$ into some model $(M', \sigma) \models TA$. By the claim, there is $\overline{\beta}_\tau \in \tilde{D}_{a'}$ such that $\sigma(\overline{\beta}_\tau) = \tau \cdot \overline{\beta}_\tau$. As $\tau \neq \mathrm{id}$, $\overline{\beta}_\tau \notin \mathrm{acl}(a')$, so $b_\tau = \pi(\overline{\beta}_\tau)$ is generic in $D_{a'}$ over $a'$. Let $\alpha : \mathrm{acl}(a'd) \cong \mathrm{acl}(a'b_\tau)$ be an elementary map extending $\mathrm{id} \restriction_{\mathrm{acl}(a')} \cup \{d \mapsto b_\tau\}$. Clearly, $\sigma' = \alpha^{-1} \circ \sigma \circ \alpha$ is an elementary permutation of $\mathrm{acl}(a'd)$ fixing $\mathrm{acl}(a') \cup \{d\}$. Moreover, if $\alpha^{-1}(\overline{\beta}_\tau) = \tau' \cdot \overline{\delta}$, then $\alpha^{-1}(\rho \cdot \overline{\beta}_\tau) = (\tau'\rho) \cdot \overline{\delta}$ for every $\rho \in G$, and an easy calculation shows that $\tau'\tau\tau'^{-1} \in H$; indeed,

$$(\alpha^{-1} \circ \sigma \circ \alpha)(\tau' \cdot \overline{\delta}) = \alpha^{-1}(\sigma(\overline{\beta}_\tau)) = \alpha^{-1}(\tau \cdot \overline{\beta}_\tau) = (\tau'\tau) \cdot \overline{\delta} = \left((\tau'\tau\tau'^{-1})\tau'\right) \cdot \overline{\delta}.$$

Thus, all conjugacy classes in $G$ are represented in $H$, showing that $H = G$ (see [HH07]). This completes the proof. $\square$

**Corollary 4.10.** *Let $T$ be stable, and assume $TA$ exists. Let $X$ be definable and $T' = \mathrm{Th}(X_{ind})$ be the theory of the induced structure on $X$. Assume $T'$ is almost $\aleph_1$-categorical.*

*Then $T'$ has the DMP. In particular, any group of finite Morley rank interpretable in $T$ (considered with the full induced structure) has the DMP.*

*Proof.* It is easy to see that if $T'_B$ has the DMP for some parameter set $B$, then $T'$ has the DMP (see [Hru92]). We may thus assume that every strongly minimal type $p'$ in $T'$ is non-orthogonal to some strongly minimal type $q'$ over $\emptyset$. Trivially, such a $q'$ is uniformly of Morley degree 1, and so $p'$ is uniformly of Morley degree 1 by Proposition 4.9. This shows that $T'$ has the DMP, by Lemma 4.7. $\square$

**Corollary 4.11.** *Let $T$ be almost $\aleph_1$-categorical, e.g. $T = \mathrm{Th}(G)$ for $G$ a group of finite Morley rank. Then $TA$ exists if and only if $T$ has the DMP.*

*Proof.* Using Fact 4.2, it is easy to see that $TA$ exists if $T$ has the DMP (see [CP98]). The other direction is the previous corollary. $\square$

# 5 Generic automorphisms of compact complex manifolds

In this section, we apply the results of Section 4 to compact complex manifolds, deducing in particular that definable groups have the DMP. We use the result of Radin [Rad04] that topological irreducibility is definable in families. Unlike in algebraic geometry, it is in general not true in compact complex manifolds that irreducibility implies Morley degree 1, so DMP does not follow immediately. Rather, we note that definability of irreducibility in the theory of a Noetherian topological structure (defined below) suffices, by a straightforward generalisation of the well-known existence of geometric axioms for ACFA, to prove axiomatisability of generic automorphisms; the results of Section 4 then apply.

We take the following definitions from [Zil10].

**Definition 5.1.**
- A *topological structure* consists of a set $S$ and a topology on each $S^n$ such that
  - (i) The co-ordinate projection maps $\mathrm{pr} : S^n \to S^m$ are continuous;
  - (ii) The inclusion maps

    $$\iota : S^m \to S^n; (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, c_{m+1}, \ldots, c_n)$$

    are continuous.
  - (iii) The diagonal $\Delta \subseteq S^2$ is closed.
  - (iv) Singletons $\{\overline{a}\} \subseteq S^n$ are closed.
- A topological structure is *Noetherian* iff the closed sets satisfy the descending chain condition.
- A topological structure is $\omega_1$-*compact* iff whenever $(C_i)_{i\in I}$ is a countable set of closed sets with the property that $\bigcap_{i\in I_0} C_i \neq \emptyset$ for any finite $I_0 \subseteq I$, then $\bigcap_{i\in I} C_i \neq \emptyset$.
- We consider a topological structure $S$ as a first-order structure in the language having a predicate for each closed set. Since singletons are closed, models $S'$ of $\mathrm{Th}(S)$ are precisely elementary extensions of $S$. We consider such $S'$ as topological structures, with closed sets the fibres with respect

to co-ordinate projections of the interpretations in $S'$ of the closed sets of $S$. It is easy to see that if $S$ is $\omega_1$-compact and Noetherian, then $S'$ is also Noetherian.

- A *constructible* set is a finite boolean combination of closed sets, so $\mathrm{Th}(S)$ has quantifier elimination iff every definable set is constructible. A constructible set is *irreducible* iff it is not the union of two relatively closed proper subsets.

- Suppose now that $T$ is the theory of an $\omega_1$-compact Noetherian topological structure $S$ with quantifier elimination.

  If $X$ is an irreducible constructible set defined over a model $S' \succcurlyeq S$, the *generic type* over $S'$ of $X$ is the complete type

  $$p_X^{S'}(x) = \{x \in O \mid O \text{ a relatively open subset of } X \text{ defined over } S'\}.$$

  Conversely, if $S'' \succcurlyeq S'$ and $a \in S''^n$, the *locus* of $a$ over $S'$ ($\mathrm{locus}(a/S')$) is the smallest closed set defined over $S'$ containing $a$.

  We say that *irreducibility is definable* in $T$ iff for any $S' \models T$ and any constructible $C(x; y) \subseteq S'^{n+m}$, the set

  $$\{y \mid C(S', y) \text{ is irreducible}\} \subseteq S'^m$$

  is definable over $S'$.

A compact complex manifold $X$ can be considered as an $\omega_1$-compact Noetherian topological structure, where the closed sets are the complex analytic subsets.

It is a result of Zilber [Zil10, Theorem 3.4.3] that the structure has quantifier elimination and finite Morley rank.

**Proposition 5.2.** *Let $T$ be the theory of an $\omega_1$-compact Noetherian topological structure with quantifier elimination, in which irreducibility is definable. Then $TA$ exists.*

*Proof.* This is a straightforward generalisation of the case of algebraically closed fields.

$TA$ is axiomatised by

(i) $(S, \sigma) \models T_\sigma$

(ii) If $U$ and $V \subseteq U \times \sigma(U)$ are closed irreducible sets in $S$ such that $V$ projects generically onto $U$ and $\sigma(U)$ (i.e. $U$ is the closure of $\mathrm{pr}_1(V)$ and $\sigma(U)$ is the closure of $\mathrm{pr}_2(V)$), and if $W \subsetneq V$ is proper closed in $V$, then there exists a point $(a, \sigma(a)) \in V \setminus W$.

By definability of irreducibility, (ii) is indeed first-order expressible.

Just as in [CH99, (1.1)], we find that $TA$ axiomatises the class of existentially closed models of $T_\sigma$; indeed: by uniqueness of generic types of irreducible sets, any existentially closed model satisfies (ii). For the converse, by quantifier elimination and closedness of equality it suffices to see that if $(S, \sigma) \models TA$ and $(S', \sigma) \models T_\sigma$ is an extension (so $S' \succcurlyeq S$), and if $V'$ is closed irreducible in $S$ and $W$ is a proper closed subset, then if there exists $a \in S'$ such that

$(a, \sigma(a)) \in V'(S') \setminus W'(S')$, then already there exists such an $a \in S$. But indeed, this follows from (ii) on taking $V := \text{locus}((a, \sigma(a))/S)$, taking $U := \text{locus}(a/S)$, and taking $W := W' \cap V$. $\square$

In particular, then

**Corollary 5.3.** *Let $T$ be the theory of a compact complex manifold. Then $TA$ exists.*

Combining with Corollary 4.10, we obtain

**Corollary 5.4.** *Let $T$ be the theory of a compact complex manifold, and suppose $T$ is almost $\aleph_1$-categorical. Then $T$ has the DMP.*

For the case of unidimensional $T$, this was deduced by a more direct method from definability of irreducibility by Radin ([Rad04]).

*Remark* 5.5. For clarity, in this section we have worked one sort at a time; but it is entirely straightforward to generalise to many-sorted topological structures, and in particular to the structure $\mathfrak{CCM}$ which has a sort for each compact complex manifold and predicates for complex analytic subsets of powers of the sorts.

*Question* 5.6. Let $(G; \cdot)$ be a Zariski group, i.e. a Noetherian Zariski structure (in the sense of [Zil10]) with a group operation $\cdot$ whose graph is closed in $G^3$. Does $G$ necessarily have definability of irreducibility (and hence, by Proposition 5.2 and Corollary 4.11, Morley degree)? What if $G$ is assumed to be presmooth?

# 6 Kummer genericity in divisible abelian groups of finite Morley

In this section, let $S$ be a definable divisible abelian group of finite Morley rank $d$ (in some stable theory $T$). In particular, since $S$ is divisible, it is connected, and so $\text{DM}(S) = 1$.

By rank considerations, the $n$-torsion subgroup $S[n]$ is finite for every $n$. We continue to denote by $T = \varprojlim_n S[n]$ the projective limit of the torsion subgroups.

Note that in general there might be a proper definable subgroup of $S$ that contains all torsion elements of $S$. By the DCC for definable subgroups in $S$, there is a smallest such group, the *definable hull* of the torsion subgroup of $S$. We denote it by $d(S_{\text{tors}})$.

We use the following version of Zilber's Indecomposability Theorem for types in commutative groups of finite Morley rank.

**Lemma 6.1.** *Let $p$ be a strong type over $A$ extending $S$. Let $p^{(d)}$ be the type of an independent $d$-tuple of realisations of $p$, and let $q := \Sigma_* p^{(d)}$ be its image under the summation map $\Sigma : S^d \to S$. Then $q$ is the generic type of an $\text{acl}^{\text{eq}}(A)$-definable coset of a connected definable subgroup $H = H(p)$ of $S$. In particular, $H(p)$ is equal to the stabiliser of $q$.*

*Proof.* It is easy to see that $n \mapsto \mathrm{RM}(\Sigma_* p^{(n)})$ is an increasing function, and that $\mathrm{RM}(\Sigma_* p^{(n)}) = \mathrm{RM}(\Sigma_* p^{(n+1)})$ implies that $\mathrm{RM}(\Sigma_* p^{(n)}) = \mathrm{RM}(\Sigma_* p^{(n+m)})$ for all $m \in \mathbb{N}$. Thus, the Morley rank of $q := \Sigma_* p^{(d)}$ is maximal among $\mathrm{RM}(\Sigma_* p^{(n)})$.

Now let $a, b$ be independent realisations of $q$, and let $c := -a - b$. Using additivity of Morley rank, we get that $a, b, c$ is pairwise independent. The result then follows from [Zie06, Theorem 1]. ☐

**Definition 6.2.** Let $p$ be a strong type extending $S$, with $H(p)$ as in Lemma 6.1.

- $p$ is called *free* if and only if $H(p) = S$;
- $p$ is called *Kummer-generic* if there is only one strong completion of the partial long type $\{[m]x_{mn} = x_n \mid m, n \in \mathbb{N}\} \cup p(x_1)$, and $p$ is called *almost Kummer-generic* if this type has only finitely many strong completions.
- for $X$ some definable subset of $S$ of Morley degree 1, we say $X$ is *free* (respectively *(almost) Kummer-generic*), if the unique generic type of $X$ is free (respectively (almost) Kummer-generic).

It follows from Lemma 6.1 that $p$ is free iff the sum of $d$ independent realisations of $p$ is generic in $S$. But note that in contrast to the case where $X$ is an irreducible subvariety of a semiabelian variety $S$, for $X$ some definable subset of $S$ of Morley degree 1, $\mathrm{RM}(\Sigma(X^d)) = d$ does not in general imply that $X$ is free.

**Lemma 6.3.** *1. Let $X \subseteq S$ be definable such that* $\mathrm{RDM}(X) = (k, 1)$.

*(a) $X$ is free iff $\{a \in S \mid \mathrm{RM}(\Sigma^{-1}(a) \cap X^d) = dk - d\}$ is a generic subset of $S$.*

*(b) $X$ is Kummer-generic iff $\mathrm{DM}([n]^{-1}(X)) = 1$ for all $n \in \mathbb{N}$.*

*2. Let $p$ be a strong type extending $S$.*

*(a) If $p'$ is a translate of a non-forking extension of $p$, then $p'$ is free (respectively (almost) Kummer-generic) iff $p$ is free (respectively (almost) Kummer-generic).*

*(b) There exists a translate $p'$ of a non-forking extension of $p$ such that $p'$ extends $H(p)$ and $\Sigma_* p'^{(d)}$ is the generic type of $H(p') = H(p)$. Moreover, the following are equivalent:*

- *$p$ is (almost) Kummer-generic;*
- *$p'$ is (almost) Kummer-generic;*
- *$H(p) \geq S_{tors}$ and $p'$ is (almost) Kummer-generic in the group $H(p)$.*

*Proof.* 1.a) Let $p$ be the generic type of $X$. Then $p^{(d)}$ is the unique generic type of $X^d$, and $X$ is free iff $\Sigma_* p^{(d)}$ is the generic type of $S$ iff for generic $a$ in $S$, $\mathrm{RM}(\Sigma^{-1}(a) \cap X^d) = dk - d$. (This uses the additivity of the Morley rank.) The result follows.

1.b) Note that since $[n]$ is finite-to-one, an element $a \in [n]^{-1}(X)$ is generic iff $na$ is generic in $X$ iff $na \models p$. From this, one may conclude.

2.a) Left to the reader.

2.b) Let $(a_1, a_2) \models p^{(2)} \mid M$. It is routine to check that $p' := \mathrm{tp}(a_2 - a_1/Ma_1)$ is as claimed. Since $p'$ is a translate of (a non-forking extension of) $p$, the first two items are equivalent by 2.a).

Now suppose $p'$ is Kummer-generic in $S$. Then, clearly $p'$ is Kummer-generic in $H(p') = H(p)$ as well. Moreover, if $\zeta \in S_{tors} \setminus H(p)$, with $n\zeta = 0$, and $a' \models p'$, there is $c' \in H(p)$ such that $[n]c' = a'$, since $H(p)$ is connected and thus divisible. Then, $\mathrm{tp}(c' + \zeta/M) \neq \mathrm{tp}(c'/M)$, since $c' + \zeta \notin H(p)$. This shows that $H(p) \geq S_{tors}$.

Conversely, assume that $H(p) \geq S_{tors}$ and $p'$ is Kummer-generic in $H(p)$. Then, for every $b \in H(p)$ and every $n \geq 1$, $[n]^{-1}(b) \subseteq H(p)$, as $S[n] \subseteq H(p)$. So Kummer-genericity of $p'$ in $S$ follows.

We now give the argument for almost Kummer-genericity. If $N \geq 1$ and $q$ is a completion of $\pi(x) := p([N]x)$, then $H(q) = H(p)$. (Indeed, in a totally transcendental divisible abelian group, if $p = [N]_* q$, it is easy to see that $\mathrm{Stab}(p) = N\,\mathrm{Stab}(q)$. Moreover, $\mathrm{Stab}(q) = N\,\mathrm{Stab}(q)$ since $\mathrm{Stab}(q)$ is connected. Thus, $\mathrm{Stab}(p) = \mathrm{Stab}(q)$.) Now $p(x)$ is almost Kummer generic iff there is $N \geq 1$ such that every completion $q$ of the partial type $\pi(x) := p([N]x)$ is Kummer-generic. So we conclude by the result for Kummer-genericity. □

**Theorem 6.4.** *Let $p$ be a strong type over $A$ extending $S$.*
*Assume $p$ is free. Then $p$ is almost Kummer-generic.*
*Moreover, $p$ is almost Kummer-generic if and only if $H(p) \geq S_{tors}$.*

*Proof.* Assume $p$ is free. Replacing $p$ by a non-forking extension if necessary, we may assume that $A = M \models T$, and that $S$ is defined over $M$.

Let $\overline{b} \models p^{(d)}$, and put $a = \Sigma(\overline{b})$. Since $p$ is free, $a$ is generic in $S$ over $M$. Let $P$ be the set of types over $\mathrm{acl}^{\mathrm{eq}}(Ma)$ extending $\mathrm{tp}(\overline{b}/Ma)$. Let $X$ be an $M$-definable set with unique generic type $p$. By Fact 4.4, Morley rank is finite and additive. It follows that $\mathrm{tp}(\overline{b}/Ma)$ is the unique type in $X^d \cap \Sigma^{-1}(a)$ over $Ma$ of maximal Morley rank, and so $P$ is equal to the set of generic types in $X^d \cap \Sigma^{-1}(a)$ over $\mathrm{acl}^{\mathrm{eq}}(Ma)$. In particular, $P$ is a finite set, say $P = \{q_1, \ldots, q_m\}$, where $m$ is the Morley degree of $X^d \cap \Sigma^{-1}(a)$.

Let $\tilde{G} = \mathrm{Gal}(Ma)$ be the (absolute) Galois group of $Ma$, i.e. the set of elementary permutations of $\mathrm{acl}^{\mathrm{eq}}(Ma)$ fixing $Ma$ pointwise. The group $\tilde{G}$ acts transitively on $P$. The proof of Claim 2 in the proof of Proposition 2.3 goes through in this context (with $\mathrm{Stab}_{\tilde{G}}(V)$ replaced by $\mathrm{Stab}_{\tilde{G}}(q_1)$), and it yields the theorem exactly as in the semiabelian case.

The “moreover” clause follows by Lemma 6.3(2)(b). □

We obtain an analogous result in the type-definable case. We refer to [BBP09, Section 2.3] for the definition of relative Morley rank.

**Theorem 6.5.** *Theorem 6.4 also holds in the case that $S$ is a type-definable divisible abelian group of finite relative Morley rank.*

*Proof.* Say $S$ is of finite relative Morley rank $d$, and let $p$ be a strong type over $A = M$ extending $S$. Assume $p$ is free (i.e. $\Sigma_* p^{(d)}$ is the generic type of $S$).

Let $X \subseteq S$ be a relatively definable subset of $S$ such that $p$ is the unique generic type of $X$. Then, for $a$ generic in $S$ over $M$, $Y = X^d \cap \Sigma^{-1}(a)$ is a relatively $Ma$-definable subset of $S^d$, so in particular the set $P$ of generic types in $Y$ over $\mathrm{acl}^{\mathrm{eq}}(Ma)$ is finite (equal to the relative Morley degree of $Y$). We finish as in the proof of Theorem 6.4. □

We now state a uniform version of Theorem 6.4.

**Theorem 6.6.** *Suppose $T$ is almost $\aleph_1$-categorical and $(S_t)_{t\in\mathcal{T}}$ is a uniformly definable family of divisible abelian groups in $T$. Let $(X_t)_{t\in\mathcal{T}}$ be a definable family of Morley degree 1 sets, with $X_t$ a free subset of $S_t$ for every $t \in \mathcal{T}$. Let $p_t$ be the generic global type of $X_t$.*

*Then there is $N \in \mathbb{N}$ such that for every $t \in \mathcal{T}$, the partial long type $\{[m]x_{mn} = x_n \mid m, n \in \mathbb{N}\} \cup p_t(x_1)$ has at most $N$ completions to global types.*

*Proof.* Since Morley rank is definable in $T$ (Fact 4.2), we may assume that $\mathrm{RM}(S_t) = d$ for all $t \in \mathcal{T}$. Let $\Sigma_t : S_t^d \to S_t$ be the summation map. We infer from Lemma 4.8 that $\mathrm{DM}(X_t^d \cap \Sigma_t^{-1}(a))$ is bounded. The statement follows, by the proof of Theorem 6.4. $\square$

**Theorem 6.7.** *Suppose $T$ is $\omega$-stable with the DMP and $(S_t)_{t\in\mathcal{T}}$ is a uniformly definable family of divisible abelian groups of finite Morley rank in $T$. Let $(X_t)_{t\in\mathcal{T}}$ be a definable family of Morley degree 1 sets, with $X_t$ a subset of $S_t$ for every $t \in \mathcal{T}$. Let $p_t$ be the generic global type of $X_t$. Then*

*(i) $\{t \in \mathcal{T} \mid p_t$ is free and Kummer-generic in $S_t\}$ is definable;*

*(ii) the set of $t$ such that some translate of $p_t$ is free and Kummer-generic in $H(p_t)$ is definable.*

*(iii) Suppose that the family $(S_t)_{t\in\mathcal{T}}$ is constant (equal to $S$). Then the set $\{t \in \mathcal{T} \mid p_t$ is Kummer-generic in $S\}$ is definable.*

*Proof.* Adding parameters to the language if necessary, we may assume the two families are defined over $\emptyset$.

(i) Using the DMP, we may assume that $\mathrm{RM}(S_t)$ is constant, say equal to $d$. We infer from Lemma 6.3 that freeness of $X_t$ (in $S_t$) is a definable condition in $t$, and so we may in addition assume that all $X_t$ are free in $S_t$. By the DMP, $\mathrm{DM}(X_t^d \cap \Sigma_t^{-1}(a))$ is bounded. As in the proof of Theorem 6.6, we find $N \in \mathbb{N}$ such that for every $t \in \mathcal{T}$, the partial long type $\{[m]x_{mn} = x_n \mid m, n \in \mathbb{N}\} \cup p_t(x_1)$ has at most $N$ completions to global types. It follows that $X_t$ is Kummer generic in $S_t$ iff $\mathrm{DM}([l]^{-1}(X_t)) = 1$ for all primes $l \leq N$. The latter is a definable condition by the DMP.

(ii) By Lemma 6.1, $H(p_t)$ is equal to the stabiliser of $\Sigma_*(p_t^{(d)})$, so $(H(p_t))_{t\in\mathcal{T}}$ is a definable family of (divisible) subgroups of the $S_t$. It is clear that one translate of $p_t$ in $H(p_t)$ is Kummer-generic in $H(p_t)$ if and only if all translates of $p_t$ in $H(p_t)$ are Kummer-generic in $H(p_t)$. We are done by part (i), since we may consider the family $\{Y_{s,t} = (s +_t X_t) \cap H(p_t) \mid \mathrm{RM}(X_t) = \mathrm{RM}(Y_{s,t}\}$.

(iii) By Lemma 6.3(2)(b), $p_t$ is Kummer generic (in $S$) iff a translate of $p_t$ is Kummer generic in $H(p_t)$ and $H(p_t) \geq S_{\text{tors}}$. The latter condition is equivalent to $H(p_t) \geq d(S_{\text{tors}})$, and so we conclude by (ii). $\square$

# 7 Interesting examples

Finally, using the results of previous sections, we give examples of groups to which the results of Section 6 apply:

*Examples* 7.1. (i) Commutative divisible algebraic groups: As mentioned in the introduction, Theorems 6.4 and 6.7 slightly generalise Theorems 1.1 and 1.2, giving results for arbitrary commutative divisible algebraic groups rather than just semiabelian varieties.

(ii) Divisible abelian groups of finite Morley rank in $DCF_0$: groups definable in $DCF_0$ have the DMP - this follows from Corollary 4.10 and the result of Hrushovski that $DCF_0A$ exists (see [Bus07]). Therefore both Theorems 6.4 and 6.7 apply.

(iii) Divisible abelian groups interpretable in compact complex manifolds: any such group is of finite MR, and by Corollary 5.4 it has the DMP. So again, Theorems 6.4 and 6.7 apply.

(iv) Type-definable groups in the theory $SCF_{p,e}$ of separably closed fields of a fixed characteristic $p$ and inseparability degree $e$: by [BD02], the commutative divisible type-definable groups in $SCF_{p,e}$ are (up to definable isomorphism) precisely those of the form $S^\# := \bigcap_n p^n S$ for $S$ a semiabelian variety.

[BBP09, Proposition 3.23] shows that for certain semiabelian varieties $S$, in particular those which are split (i.e. a product of a torus and an abelian variety), $S^\#$ has finite relative MR. Hence Theorem 6.5 applies.

It is known that $SCF_{p,e}A$ exists ([Cha01]), so it is tempting in the context of the last example to try to deduce DMP for $S^\#$ by appeal to Corollary 4.10, for those $S^\#$ having relative quantifier elimination; however, it is not clear that existence of $SCF_{p,e}A$ implies axiomatisability of the generic automorphism for the induced structure on $S^\#$. In an earlier version of the paper, we put a general version of this as a question, asking:

*Question.* If $T$ is a stable theory for which $TA$ exists, and if $X$ is a type-definable set with relative QE, does $Th(X_{ind})A$ necessarily exist?

The general answer to this question is no, as is shown by the following example due to Nick Ramsey.

*Example* 2. Let $\mathcal{L}$ be the language consisting of a binary relation $E$ and constant symbols $c_{i,n}$ for $1 \leq i \leq n < \omega$. Let $T$ be the $\mathcal{L}$-theory axiomatised by:

- $E$ is an equivalence relation with infinitely many classes, and every class is infinite;
- the $c_{i,n}$ are pairwise distinct, and $E(c_{i,n}, c_{j,m})$ holds iff $n = m$.

$T$ is complete with QE, has finite additive Morley rank and the DMP. So $TA$ exists (see [CP98]).

For $n \geq 1$, consider $\varphi_n(x) := E(x, c_{n,n}) \to \bigvee_{i \leq n} x = c_{i,n}$, and let $X$ be the type-definable set given by $\{\varphi_n \mid n \geq 1\}$. The induced structure on $X$ is that of an equivalence relation with exactly one equivalence class for every $n \geq 1$ whose elements are named by constants. In particular, $X$ has relative QE and $X_{ind}$ is

stable with the finite cover property. By an observation of Kudaibergenov (see [Kik00, Fact 3.5]), it follows that $Th(X_{ind})A$ does not exist.

In a similar fashion, Ramsey obtains an example where $X$ has relative QE, $X_{ind}$ is nfcp but $Th(X_{ind})A$ does not exist.

Finally, let us mention (see the following remark) that there is another way of viewing the notion of Kummer-genericity for subvarieties of a semiabelian variety in a more abstract manner. This reformulation shows that the main ideas of the present paper are not bound to the presence of an underlying (commutative) group. It is then reasonable to ask whether our results extend to more general expansions of $\omega$-stable (finite rank) theories, e.g. in the context of Shimura varieties. In the model-theoretic study of the $j$-function, the relationship between affine $n$-space (with its structure as an algebraic variety) and its reduct given by the Hecke correspondences plays an important role, and there are strong similarities to the semiabelian context. Forthcoming work of Adam Harris is expected to be relevant to this case.

*Remark* 7.2. Suppose $S$ is a semiabelian variety defined over $K = K^{alg}$. Let $T_1$ be the theory of the structure $S_1$ with underlying set $S(K)$, and with a predicate for every algebraic subvariety (defined over $K$) of some cartesian power of $S$. Let $T_0$ be the reduct of $T_1$, with predicates only for algebraic subgroups of cartesian powers of $S$. Let $T_i$ be $\mathcal{L}_i$-theories, for $i = 0, 1$, and put $S_0 = S_1 \upharpoonright_{\mathcal{L}_0}$. (Note that $S_0$ is an abelian structure, so in particular $T_0$ is one-based.)

Let $X \subseteq S$ be an irreducible variety defined over $L = L^{alg} \supseteq K$, and let $p_1$ be the generic $\mathcal{L}_1$-type of $X$ over $S(L)$. Then $X$ is free iff $p_1 \upharpoonright_{\mathcal{L}_0}$ is the (unique) generic $\mathcal{L}_0$-type over $S(L)$, and $X$ is Kummer-generic iff it is free and for $b$ realising $p_1$, the natural map of absolute Galois groups $\mathrm{Gal}_{T_1}(Lb) \to \mathrm{Gal}_{T_0}(Lb)$ is a surjection.

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